PLASTIC COLLAPSE OF 3D ALUMINIUM FRAMES VIA LIMIT AND SHAKEDOWN ANALYSIS

Alexios T. Ampatzis
Dr. Civil Engineer
Institute of Metal Structures, Department of Civil Engineering
Aristotle University of Thessaloniki Thessaloniki, GR 54124, Greece
e-mail: aabatzis@gmail.com

Vasileios G. Psomiadis
Dr. Civil Engineer
Institute of Metal Structures, Department of Civil Engineering
Aristotle University of Thessaloniki Thessaloniki, GR 54124, Greece
e-mail: vpsomiadis@gmail.com

Evangelos Efthymiou
Lecturer, Dr Civil Engineer
Institute of Metal Structures, Department of Civil Engineering
Aristotle University of Thessaloniki Thessaloniki, GR 54124, Greece
e-mail: vefth@civil.auth.gr

1. SUMMARY

In the field of plastic design of metal structures, the Limit and Shakedown Analysis (LiSA) approach has been widely exploited to determine the plastic collapse load capacity, as it provides advantages in terms of computational robustness in comparison with incremental non-linear analysis. However, aside to its effectiveness LiSA is mostly applied to steel, characterized by rigid-perfectly- plastic behavior, while limited research on its exploitation in case of hardening/ softening materials is registered among the scientific community. The present work aims to investigate a LiSA formulation suitable for the plastic limit load assessment in the case of 3D aluminium frame structures. In contrast with steel, aluminium's post elastic behavior is characterized by a semantic strain hardening feature, as well as by its bounded ductility, influencing and limiting the development of the collapse mechanism. For the purposes of this study, the analytical framework of the methodology is highlighted and the feasibility of its implementation through deployment of linearized approximations to codified failure criteria and hardening constitutive models for aluminium, provided by Eurocode 9, are investigated.

2. INTRODUCTION

Metal and especially steel structures, given their broad plastic deformation capacity, have been often at the core of research for the evaluation of the post elastic behavior via the direct methods of plasticity. In recent years, aluminium based alloys have found increased structural applications as a modern construction material with architectural appeal and special characteristics [1] leading to the development of relevant scientific research [2,3] along with the European design code “Eurocode 9” [4]. As aluminium is characterized by a semantic strain-hardening feature as well as by its bounded ductility, exploiting the full potential of the material requires the consideration of its work-hardening behavior.

LiSA approaches that can determine the plastic collapse load of the structure independently of the exact load history, have been proposed for various yield criteria, stress states and constitutive models [5,6]. As consideration of kinematic hardening is important for many engineering problems several attempts have been made on extending the classic Melan-Koiter theorems to various types of nonlinear hardening behavior. Unlimited kinematical hardening was soon proved insufficient for incremental collapse (ratcheting) [7] and research shifted to limited kinematical hardening [8,9,10,12,3]. In particular at [8] a simplified two-surface model was introduced which was also used in [3,12] for a lower and an upper bound approach respectively. Almost all the above implementations involve stress and strain tensors of the material point and the von Mises yield criterion.

In this paper, a formulation based on Melan’s lower bound theorem is presented for the LiSA of bounded kinematic hardening aluminium frame 3D structures with Eurocode 9’s combined biaxial bending and axial force criterion. An alternative 2-surface plasticity model is proposed for linearized local failure criteria in the space of generalized cross section stresses, which effectively addresses problems containing non-Euclidian-norm constraints. A numerical example of a multi-storey building is included, utilizing FEM and the 3-node Timoshenko column-beam element, to validate the proposed formulation and to study the influence of hardening effect.

3. THEORY

Let Ω be a 3D spatial aluminium frame, discretized by the Finite Element Method (FEM) into n_E column-beam finite elements and n_G Gauss points. Let also $\boldsymbol{L}^F$ be a convex polytope of external loads with n_F vertices, applied on Ω. The loading domain $\boldsymbol{L}^F$ is comprised of a constant loading $\boldsymbol{v}_0$, the central part, and a variable loading $\alpha\boldsymbol{v}^i$, α being a non-negative scalar. The elastic stress-resultant vectors $\boldsymbol{\sigma}^i$ of the form $\{N, V_y, V_z, M_x, M_y, M_z\}$ are the respective FEM elastic analysis results to the loading vectors $\boldsymbol{v}_0$ and $\alpha\boldsymbol{v}^i$, such that:

$$\boldsymbol{v}_0 + \alpha\boldsymbol{v}^i \in \boldsymbol{L}^F \text{ and } \boldsymbol{\sigma}^i = \alpha\boldsymbol{s}^i + \boldsymbol{s}_0,\ i=1,\dots,n_F \tag{1}$$

The total elastoplastic stress-resultant vectors $\boldsymbol{u}^i$, which are the superposition of $\boldsymbol{\sigma}^i$ and a self equilibrating stress field $\boldsymbol{\rho}$, are bounded by the convex set of the local failure criteria $\boldsymbol{F}_{,j}$, $j=1,\dots,n_G$. In this framework, the classical Melan's lower bound shakedown theorem denotes that there exists a unique solution $\{\alpha, \boldsymbol{\rho}\}$ for which the structure's plastic work is bounded – shakedown stress state. The Shakedown Analysis (*SDA*) problem is written as:

$$P_1(\alpha_{SDA}, \boldsymbol{\rho}): \max \alpha, \quad \text{s.t.:} \tag{2}$$

$$\boldsymbol{u}^i = \boldsymbol{\sigma}^i + \boldsymbol{\rho} \in \{\boldsymbol{F}_{,j}\},\ i=1,\dots,n_F,\ j=1,\dots,n_G$$

$$\mathbf{H}\,\boldsymbol{\rho} = \mathbf{0}$$

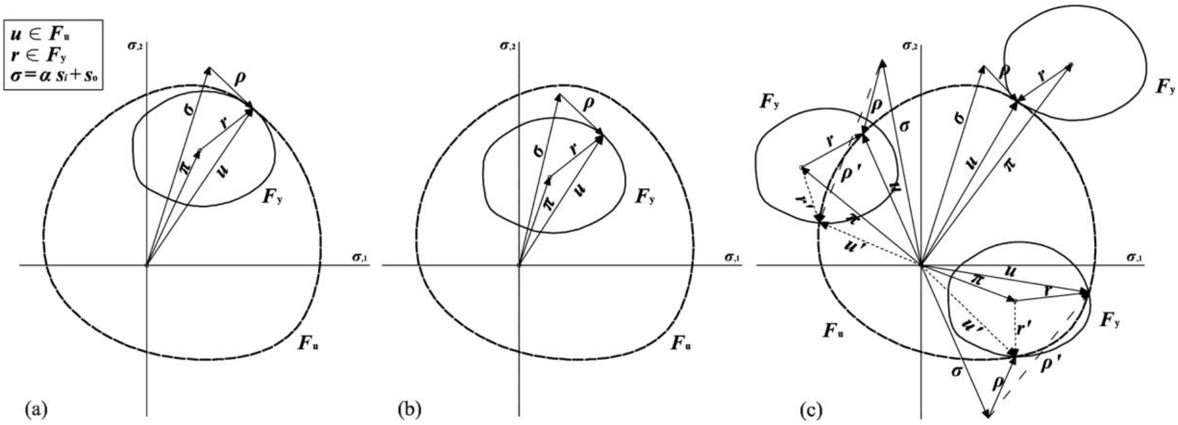
H is the equilibrium matrix of the structure and equality $\mathbf{H}\,\boldsymbol{\rho} = \mathbf{0}$ represents the null space condition of the problem. The Limited Analysis (*LA*) problem is a special case of problem P_1 where the loading domain shrinks to a singleton, i.e. $n_F=1$. Problem P_1 can be formed as a classical mathematical programming problem and its solution depends on the type/form of the failure criteria inequalities.

Let $\{\boldsymbol{F_y}^j\}$ be the set of local yield criteria limiting the elastic region and $\{\boldsymbol{F_u}^j\}$ the set of local failure criteria defining the ultimate bearing capacity of the cross sections (pic. 1a & 1b). Keeping the previous notation of P_1 then the *SDA* problem for structures with limited kinematic hardening is written as:

$$P_2\,(\alpha_{SDA.LKH}, \boldsymbol{\rho}, \boldsymbol{\pi})\text{: max } \alpha,\ \text{ s.t.:} \tag{3}$$
$$\boldsymbol{u}^i = \boldsymbol{\sigma}^i + \boldsymbol{\rho} \ \in \{\boldsymbol{F_u}^j\},\ i=1,\dots,n_F,\ \ j=1,\dots n_G$$
$$\boldsymbol{r}^i = \boldsymbol{\sigma}^i + \boldsymbol{\rho} - \boldsymbol{\pi} \ \in \{\boldsymbol{F_y}^j\},\ i=1,\dots,\ n_F,\ \ j=1,\dots n_G$$
$$\mathbf{H}\,\boldsymbol{\rho} = \mathbf{0}$$

where, the vectors $\boldsymbol{\pi}$ called the backstresses, define the bouncing of the yield surface $\boldsymbol{F}_{\mathbf{y}}$ into the ultimate region $\boldsymbol{F_u}$.

It is noted that the limited kinematic hardening theory allows the free transposition of the yield surface $\boldsymbol{F_y}$ without rotation or shape alteration, as long as it remains inside the bounds of the failure domain $\boldsymbol{F_u}$. Up to date and to the extent of the writers' knowledge the *SDA* formulation for limited kinematic hardening has been applied for the von-Mises criterion or Mises-type loci defining the yield and ultimate failure surfaces, which are Euclidian norms on the stress deviators (second order/length norms) [3], [9], [10]. In the case of length norms are used as failure criteria, the conditions of P_2 are adequate to restrict the movement of $\boldsymbol{F_y}$ in the domain $\boldsymbol{F_u}$, due the triangle inequality theorem (pic. 1a & 1b). In this work, convex hyperpolyhedra are used as local failure criteria, which are produced as the piecewise linear approximation of complex higher order surfaces describing the failure of aluminium hollow sections.



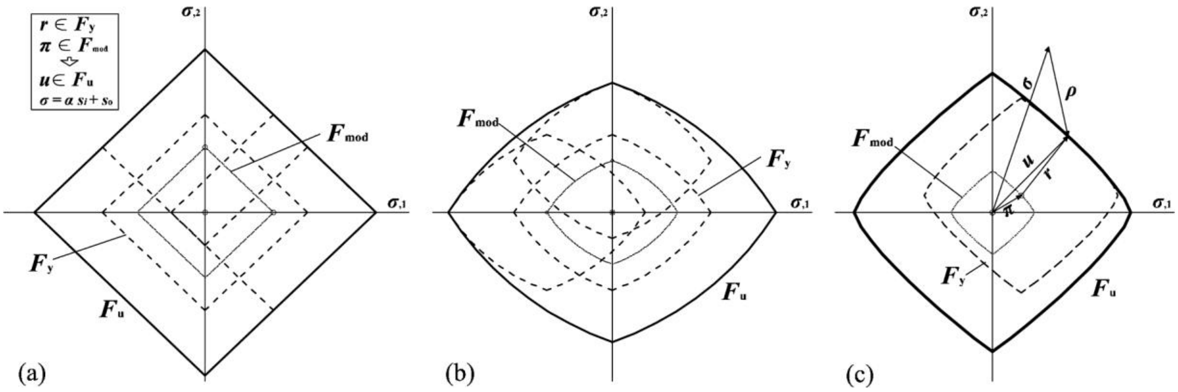
Pic. 1 The shakedown analysis problem for limited kinematic hardening

In this framework, where sets of linear inequalities are used to describe the failure surfaces, the case described in pic. 1c arises, which leads to the “relaxation” of the limited kinematic hardening constraints and possibly into the lack of the uniqueness of the mathematical programming solution. To address this drawback of P_2, a modified formulation of the *SDA limited kinematic hardening* problem containing only linear inequalities is proposed:

$$P_3\,(\alpha, \boldsymbol{\rho}, \boldsymbol{\pi}):\ \max \alpha,\ \text{s.t.:} \tag{4}$$
$$\boldsymbol{\pi} \in \{\boldsymbol{F}_{\mathbf{mod}}^{\ j}\},\ j=1,...,n_G$$
$$\boldsymbol{r}^i = \boldsymbol{\sigma}^i + \boldsymbol{\rho} - \boldsymbol{\pi} \in \{\boldsymbol{F}_y^{\ j}\},\ i=1,...,n_F,\ j=1,...,n_G$$
$$\mathbf{H}\,\boldsymbol{\rho} = \mathbf{0}$$

Surface $\boldsymbol{F}_{mod}$ is defined as a homothetic transformation of the failure surface $\boldsymbol{F}_u$. In essence, limiting the backstress vectors $\boldsymbol{\pi}$ by the surfaces $\boldsymbol{F}_{mod}$, see pic. 2. The proposed modified surface bounds the movement of the origin of the yield surface $\boldsymbol{F}_y$ such that:

$$\{\boldsymbol{F}_y^{\ j} \cup \boldsymbol{F}_{mod}^{\ j}\} \subseteq \boldsymbol{F}_u^{\ j} \tag{5}$$

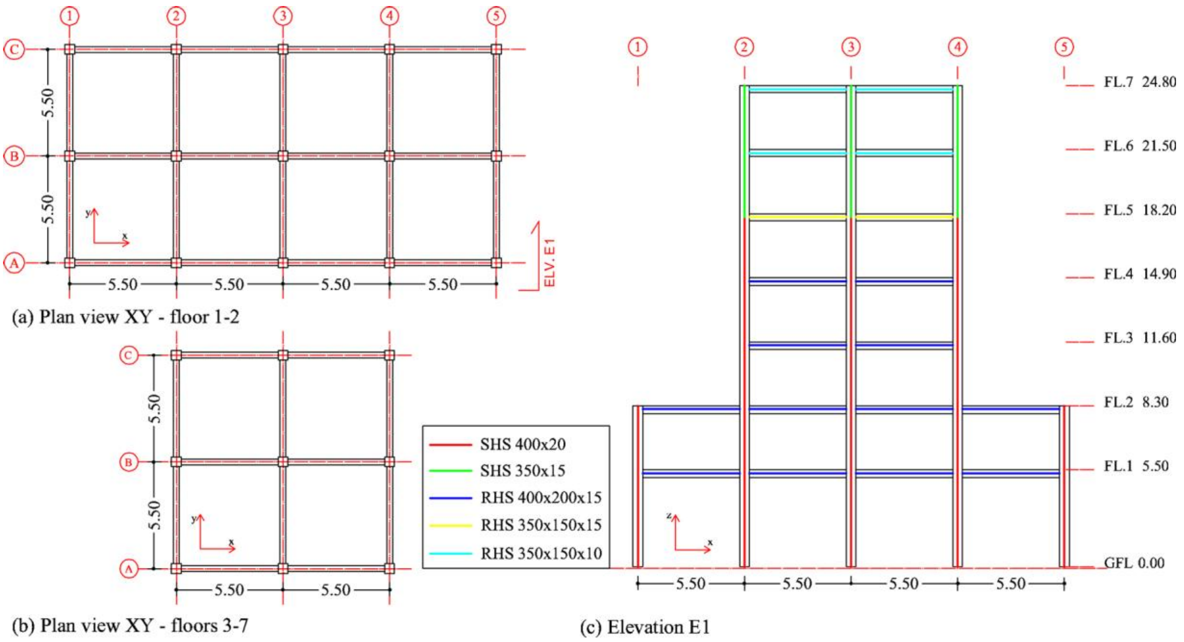


Pic. 2 Proposed modified SDA formulation for limited kinematic hardening with linearized criteria

It can be seen that the size of the modified *SDA* problem P_3 is drastically reduced compared to P_2 , as the inequalities $\boldsymbol{u}^i \in \{\boldsymbol{F}_u^{\ j}\}$ are replaced by the fewer $\boldsymbol{\pi} \in \{\boldsymbol{F}_{mod}^{\ j}\}$. It should be noted that both the yield surface, $\boldsymbol{F}_\mathbf{y}$, and the modified surface, $\boldsymbol{F}_{mod}$, must be produced from $\boldsymbol{F}_u$ by homothecy from $\boldsymbol{F}_u$. Deviation to this principle might not be unacceptable and shall be investigated in a future work.

4. EXAMPLE

The numerical example concerns a vertically irregular 7-storey 3D Moment Resisting (MRF) aluminium frame. The configuration and dimensions are described in pic. 3. The structural members are orthogonal extruded tubes (exact profile assignments shown in pic. 3) and the material selected is the aluminium alloy 6082-T6. According to EC9 the selected closed perimeter sections assure class 1 and exhibit extended post-elastic deformation capability thus allowing for full plastic design. At each floor level, diaphragm constraints are imposed in order to ensure solid disk movement. The gravitational loads of the structure, which comprise the central part of the loading, are: dead 2.9kN/m^2 and imposed 3.5kN/m^2. The lateral part of the loading, which is considered variable, is wind pressure - calculated according to EC1 - along the two principal directions of the frame (wind x-x, wind y-y). The resultant forces of the upwind and downwind pressure, $\boldsymbol{W}_\text{x}$ and $\boldsymbol{W}_\text{y}$, are applied to the centroids of each floor (Table 1), i.e. the master joints of each diaphragm.



(a) Plan view XY - floor 1-2

(b) Plan view XY - floors 3-7

(c) Elevation E1

Pic. 3 Seven storey spatial aluminium moment resisting frame & section assignments.

The frame is discritized with FEM using the 3-node Timoshenko element containing 2-Gauss points, which is capable of capturing shear virtual work along with bending [13]. The design load combinations for ULS and SLS are considered, according to EC1, and the sections are verified with the EC9 guidelines. The decisive design checks are PMM strength checks, for ULS, and the control of horizontal drifts of the floors for SLS. It has to be noted that the relatively slack limit of H/300, used in the limitation of lateral displacements, is marginally satisfied by the behavior of the structure under review.

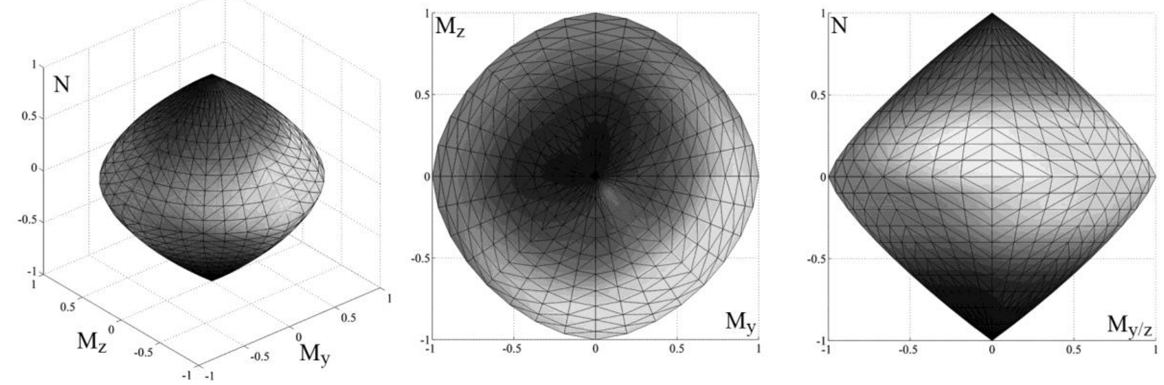
Floor	**Fl. 1**	**Fl. 2**	**Fl. 3**	**Fl. 4**	**Fl. 5**	**Fl. 6**	**Fl. 7**
	kN	*kN*	*kN*	*kN*	*kN*	*kN*	*kN*
$\boldsymbol{W}_x$	65,51	43,24	43,24	62,26	62,26	62,26	62,26
$\boldsymbol{W}_y$	125,78	83,01	43,24	62,26	62,26	62,26	62,26

Table 1. Lateral wind forces of directions x-x, y-y

The material softening due to welds near the moment resisting connections, HAZ, is assessed in the design and analysis of the aluminium frame by the introduction of reduction factors for the yield $\rho_{o.haz}\, f_o$ and ultimate $\rho_{u.haz}\, f_u$ strengths.

The elastic-plastic non-linear behavior is modeled by the plastic hinge method. The proposed two surface plasticity model described in Section 3, is assigned to the plastic hinges that coinside with the Gauss points FEM grid. Column behavior is governed by the PMM interaction criterion of Eq. 6.2.9.2. of EC9. The linearization of the criterion produces a set of 1216 facets in the space of N, M_y, M_z (pic. 4), inscribed in the convex hull of the PMM failure surface. Then the linearized hull is scaled by using the respective capacities of the limit state under question.

The stress-component capacities for each limit state are calculated based on EC9 guidelines. The respective HAZ affected values are produced by multiplying with the reduction factors, which for alloy 6082-T6 are $\rho_{o.haz} = 0{,}42$ and $\rho_{u.haz} = 0{,}48$.



Pic. 4 EC9 Linearized PMM interaction failure surface – plastic hinge for aluminium columns.

Non-linear analyses are run with the variable load vectors $\boldsymbol{W}_\mathrm{x}$ and $\boldsymbol{W}_\mathrm{y}$ following the methodology of *LA* and *SDA* problems P_1 and P_3, and the variable load domain multipliers, α, are calculated. Three series of analysis cases are produced using a) the elastic-perfect plastic model with the yield surface $\boldsymbol{F}_y$, b) the elastic-perfect plastic model with the ultimate surface $\boldsymbol{F}_u$ and c) the proposed 2-surface methodology ($\boldsymbol{F}_\mathrm{y}$, $\boldsymbol{F}_{mod}$), in order to measure the effect of limited kinematic hardening on the safety factors.

Problem	$LA_{(a)}$	$LA_{(b)}$	$LA_{(c)}$	$P_{1(a)}^{SDA\text{-}cyc}$	$P_{1(b)}^{SDA\text{-}cyc}$	$P_{3(c)}^{SDA\text{-}cyc}$	$P_{1(a)}^{SDA\text{-}tri}$	$P_{1(b)}^{SDA\text{-}tri}$	$P_{3(c)}^{SDA\text{-}tri}$
	α_{LA}	α_{LA}	$\alpha_{LA.LKH}$	$\alpha_{SDA\text{-}cyc}$	$\alpha_{SDA\text{-}cyc.}$	$\alpha_{SDA.LKH\text{-}cyc}$	$\alpha_{SDA\text{-}tri}$	$\alpha_{SDA\text{-}tri}$	$\alpha_{SDA.LKH\text{-}tri}$
$\boldsymbol{W}_x$	4,13	5,40	5,40	3,21	4,15	**3,21**	-	-	-
$\boldsymbol{W}_y$	3,49	4,57	4,57	2,89	3,82	**2,95**	-	-	-
$\boldsymbol{W}_x + \boldsymbol{W}_y$	3,19	4,32	4,32	-	-	-	3,48	4,56	**4,52**

Table 2. Plastic load multipliers for LA & SDA problems with/without limited kinematic hardening

In the inelastic analysis cases contained in Table 2 three shapes of loading domains are included. For *LA* it is the obvious single-point set, $\boldsymbol{L}^{(1)}$. For *"SDA-cyc"* $\boldsymbol{L}^{(2)}$ is a uniaxial, symmetric-about zero, set [-$\boldsymbol{W}_\mathbf{x}$; $\boldsymbol{W}_\mathbf{x}$] or [-$\boldsymbol{W}_y$; $\boldsymbol{W}_y$]. In the *"SDA-tri"* case the variable load domain $\boldsymbol{L}^{(3)}$ is a triangle set [0; $\boldsymbol{W}_\mathbf{x}$; $\boldsymbol{W}_y$]. It is noted that the effect of limited kinematic hardening on the safety multipliers is limited to the *SDA* cases, see problems $P_{3(c)}^{SDA\text{-}cyc}$ and $P_{3(c)}^{SDA\text{-}tri}$ compared to problems $P_{1(b)}$ - rigid plastic material with ultimate stress f_u (Table 2). It is rather interesting that for loading $\boldsymbol{W}_x$ low cycle fatigue failure arises.

4. COMMENTS - CONCLUSION

A new two-surface methodology is proposed to simulate the limited kinematic hardening of aluminium 3D frames, via the direct methods of plasticity with linearized yield/failure criteria. The framework presented makes use of a new modified surface parallel to the ultimate bounding surface, to overcome the classical two-surface model's weakness when the constraints are not length norms. The numerical example shows the applicability of the proposed methodology along with the effectiveness and robustness for LiSA on aluminium frame structures, which exhibit semantic post elastic features.

The FEM modeling and elastic analyses were carried out by a FEM research code [13] implementing the 3D Timoshenko 3-node column-beam element, diaphragm constraints and capable of exporting the **H** matrices. The mathematical programming problems were input to MOSEK [14] with the aid of linking Matlab utilities built for the purpose. Table 3 presents the sizes of the problems solved.

Problem	LA	LA^{LKH}	$P_1^{SDA\text{-}cyc.}$	$P_3^{SDA\text{-}cyc.}$	$P_1^{SDA\text{-}tri}$	$P_3^{SDA\text{-}tri}$
Unknowns	10021	20041	10021	20041	10021	20041
Constraints	189904	373424	373424	556944	556944	740464

Table 3. Size, unknowns and constraints, of LA and SDA problems solved

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ΠΛΑΣΤΙΚΗ ΚΑΤΑΡΡΕΥΣΗ ΧΩΡΙΚΩΝ ΠΛΑΙΣΙΩΝ ΑΛΟΥΜΙΝΙΟΥ ΜΕΣΩ ΟΡΙΑΚΗΣ ΑΝΑΛΥΣΗΣ ΚΑΙ ΑΝΑΛΥΣΗΣ ΠΡΟΣΑΡΜΟΓΗΣ

Αλέξιος Θ. Αμπατζής
Δρ. Πολιτικός Μηχανικός
Εργαστήριο Μεταλλικών Κατασκευών, Τμήμα Πολιτικών Μηχανικών,
Αριστοτέλειο Πανεπιστήμιο Θεσσαλονίκης, Θεσσαλονίκη, ΤΚ 54124, Ελλάδα
e-mail: aabatzis@gmail.com

Βασίλειος Γ. Ψωμιάδης
Δρ. Πολιτικός Μηχανικός
Εργαστήριο Μεταλλικών Κατασκευών, Τμήμα Πολιτικών Μηχανικών,
Αριστοτέλειο Πανεπιστήμιο Θεσσαλονίκης, Θεσσαλονίκη, ΤΚ 54124, Ελλάδα
e-mail: vpsomiadis@gmail.com

Ευάγγελος Ευθυμίου
Λέκτορας, Δρ. Πολιτικός Μηχανικός
Εργαστήριο Μεταλλικών Κατασκευών, Τμήμα Πολιτικών Μηχανικών,
Αριστοτέλειο Πανεπιστήμιο Θεσσαλονίκης, Θεσσαλονίκη, ΤΚ 54124, Ελλάδα
e-mail: vefth@civil.auth.gr

ΠΕΡΙΛΗΨΗ

Στον πλαστικό σχεδιασμό μεταλλικών κατασκευών η οριακή ανάλυση και η ανάλυση προσαρμογής (LiSA) έχουν χρησιμοποιηθεί εκτενώς για τον προσδιορισμό του πλαστικού φορτίου κατάρρευσης καθώς παρέχουν το πλεονέκτημα της υπολογιστικής αμεσότητας συγκριτικά με την μη-γραμμική βηματική ανάλυση. Ωστόσο, πέρα από την αποτελεσματικότητά τους, ως επί το πλείστον βρίσκουν εφαρμογή στον χάλυβα, που χαρακτηρίζεται από τέλεια ελαστοπλαστική συμπεριφορά ενώ καταγράφεται περιορισμένη έρευνα σχετικά με την εκμετάλλευσή τους σε υλικά που παρουσιάζουν έντονο το φαινόμενο της κράτυνσης. Η παρούσα εργασία στοχεύει στην διερεύνηση μιας διατύπωσης LiSA κατάλληλης για την εύρεση του πλαστικού φορτίου κατάρρευσης κατασκευών αποτελούμενων από τριδιάστατα πλαίσια αλουμινιού. Σε αντίθεση με τον χάλυβα η μεταλεστική συμπεριφορά του δομικού αλουμινίου, η οποία χαρακτηρίζεται από συνεχή κράτυνση καθώς και μειωμένη διαθέσιμη πλαστιμότητα, επηρεάζει σημαντικά την εξέλιξη του μηχανισμού κατάρρευσης. Για τους σκοπούς της παρούσας μελέτης επισημαίνεται το αναλυτικό πλαίσιο της προτεινόμενης μεθοδολογίας και παράλληλα εξετάζεται η δυνατότητα εφαρμογής της μέσω της ανάπτυξης γραμμικοποιημένων προσεγγίσεων σε κριτήρια αστοχίας και κρατυνόμενο καταστατικό νόμο υλικού όπως παρέχονται ειδικά για το αλουμίνιο από τον Ευρωκώδικα 9.